MATH2050C Assignment 12

Section 5.6 no. 3, 4, 11.

Deadline: April 15 , 2025.

Hand in: 5.6 no. 3, 4, 11.

Supplementary Problems

1. (Optional) Order the rational numbers in $(0,1)$ into a sequence $\{x_k\}$. Define a function on $(0,1)$ by $\varphi(x) = \sum 1/2^k$ where the summation is over all indices k such that $x_k < x$. Show that

   (a) φ is strictly increasing and $\lim_{x\to 1^-} \varphi(x) = 1$.

   (b) φ is discontinuous at each x_k.

   (c) φ is continuous at each irrational number in $(0,1)$.

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The Exponential Function

We study the exponential function, and its inverse function namely the logarithmic function. Then we use it to define the power functions.

Define

$$E(x) = \lim_{n\to\infty} \left(1+\frac{x}{n}\right)^n .$$

Previously we have established the following facts:

- The limit $E(x) \equiv \lim_{n\to\infty}(1+x/n)^n$ exists for all $x\in\mathbb{R}$.
- For $x \geq 0$, the sequence $\{(1+x/n)^n\}$ is increasing.
- For $x\in\mathbb{R}, E(x)E(-x) = 1$.
- For $x \geq 0, E(x) = \sum_{k=0}^\infty x^k/k!$.

Now we establish more.

Theorem 12.1 For $x \in \mathbb{R}$,

$$E(x) = \sum_{k=0}^\infty \frac{x^k}{k!} .$$

Proof Let $x\in[-a,a]$, we first claim there are k_0 and M depending only on a so that

$$\frac{|x|^k}{k!} \leq \frac{M}{2^k}, \quad k \geq k_0+1. \tag{1}$$

For, fixing k_0 so that $a/(k_0+1) \leq 1/2$, we have

$$\begin{aligned}
\frac{|x|^k}{k!} &\leq \frac{a^k}{k!} \\
&= \frac{a}{1}\frac{a}{2}\cdots\frac{a}{k_0}\frac{a}{k_0+1}\cdots\frac{a}{k} \\
&< \frac{a}{1}\frac{a}{2}\cdots\frac{a}{k_0}\frac{1}{2^{k-k_0}} \\
&= \frac{M}{2^k}, \quad M = \frac{(2a)^{k_0}}{k_0!}.
\end{aligned}$$

For $k \geq k_0+1$ and large n, we have

$$\begin{aligned}
\left|\left(1+\frac{x}{n}\right)^n - \sum_{j=0}^k C_j^n \frac{x^j}{n^j}\right| &= \left|\sum_{j=k+1}^n C_j^n \frac{x^j}{n^j}\right| \\
&\leq \sum_{j=k+1}^n \frac{|x|^j}{j!} \\
&\leq \sum_{j=k+1}^n \frac{M}{2^j} \\
&\leq \frac{M}{2^{k+1}} \sum_{j=0}^\infty \frac{1}{2^j} \\
&= \frac{M}{2^k} ,
\end{aligned}$$

after using (1). Noting that

$$C_j^n \frac{x^j}{n^j} = \Big(1 - \frac{1}{n}\Big)\Big(1 - \frac{2}{n}\Big) \cdots \Big(1 - \frac{j-1}{n}\Big)\frac{x^j}{j!}.$$

Letting $n \to \infty$, we have

$$\Big|E(x) - \sum_{j=0}^{k} \frac{x^j}{j!}\Big| \le \frac{M}{2^k} \ .$$

Given $\varepsilon > 0$, choose K such that $M/2^K < \varepsilon$. For $k \ge K$, $|E(x) - \sum_{j=0}^k x^j/j!| \le M/2^k \le M/2^K < \varepsilon$. The conclusion follows.

Theorem 12.2 For $x, y \in \mathbb{R}$, $E(x+y) = E(x)E(y)$.

Proof Assume $x, y > 0$ first. (If one of x, y is zero, the identity holds obviously.)

$$\begin{aligned}\Big(1 + \frac{x}{n}\Big)\Big(1 + \frac{y}{n}\Big) &= \Big(1 + \frac{x+y}{n} + \frac{xy}{n^2}\Big) \\ &= \frac{\Big(1 + \frac{x+y}{n} + \frac{xy}{n^2}\Big)}{\Big(1 + \frac{x+y}{n}\Big)} \times \Big(1 + \frac{x+y}{n}\Big) \ .\end{aligned}$$

Using $1 \le \frac{1+a+b}{1+a} \le 1 + b$ for $a, b \ge 0$, we have

$$1 \le \left[\frac{\Big(1 + \frac{x+y}{n} + \frac{xy}{n^2}\Big)}{\Big(1 + \frac{x+y}{n}\Big)}\right]^n \le \Big[\Big(1 + \frac{xy}{n^2}\Big)\Big]^n \le E(xy)^{1/n} \ .$$

In the last step we have used the fact that $(1 + xy/n)^n$ increases to $E(xy)$, hence $(1 + xy/n)^n \le E(xy)$ and $(1 + xy/n^2)^n \le E(xy)^{1/n}$. Using $a^{1/n} \to 1$ as $n \to \infty$ for any $a > 0$, we conclude by the Squeeze Theorem that

$$\lim_{n \to \infty} \left[\frac{\Big(1 + \frac{x+y}{n} + \frac{xy}{n^2}\Big)}{\Big(1 + \frac{x+y}{n}\Big)}\right]^n = 1 \ .$$

Therefore, by passing limit in

$$\Big(1 + \frac{x}{n}\Big)^n \Big(1 + \frac{y}{n}\Big)^n = \left[\frac{\Big(1 + \frac{x+y}{n} + \frac{xy}{n^2}\Big)}{\Big(1 + \frac{x+y}{n}\Big)}\right]^n \times \Big(1 + \frac{x+y}{n}\Big)^n \ ,$$

we conclude $E(x)E(y) = E(x+y)$ for $x, y \ge 0$. The remaining cases are (a) $x > 0, y < 0$ and $x + y > 0$, (b) $x > 0, y < 0, x + y < 0$, and (b) $x, y < 0$. For (a), $E(-y)E(x+y) = E(x)$ holds. Using $E(-y)E(y) = 1$, $E^{-1}(y)E(x+y) = E(x)$ which is $E(x+y) = E(x)E(y)$. Cases (b) and (c) can be deduced in a similar way.

Theorem 12.3 $E(x)$ is strictly increasing and $\lim_{n \to \infty} E(x) = \infty$ and $\lim_{n \to -\infty} E(x) = 0$.

Proof For $x > 0$, from Theorem 12.1, $E(x) > 1+x > 1$. For $y > x > 0$, $E(y) = E(x+y-x) = E(x)E(y-x) > E(x)$ since $E(y-x) > 1$. Hence E is strictly increasing on $[0, \infty)$. Using the relation $E(x) = 1/E(-x)$, E is also strictly increasing on $(-\infty, 0)$. We conclude that E is strictly increasing on $\mathbb{R}$. Finally, from $E(x) \geq 1+x$, we see that $E(x)$ diverges as $x \to \infty$. On the other hand, using $E(x) = (E(-x))^{-1}$, $E(x)$ decays to 0 as $x \to -\infty$.

Theorem 12.4 $E(x)$ is continuous everywhere.

Proof First we claim that E is continuous at $x = 0$. For $x \in [-1, 1]$,

$$|E(x) - 1| = \left| x \left(1 + \frac{x}{2!} + \frac{x^2}{3!} + \frac{x^3}{4!} + \cdots \right) \right| \leq E(1)|x|.$$

Therefore, $\lim_{x \to 0} E(x) = 1 = E(0)$. Hence E is continuous at $x = 0$. At an arbitrary c, $E(c+h) - E(c) = E(c)(E(h) - 1) \to 0$ as $h \to 0$, so E is continuous at c.

Here we present a more general result which shows not only the exponential functions, but also the sine and cosine functions are continuous everywhere.

Theorem 12.5 Consider the infinite series

$$s(x) = \sum_{k=0}^{\infty} a_k \frac{x^k}{k!} \ ,$$

where $a_k \in \{-1, 0, 1\}$ for all k. Then s is continuous on $(-\infty, \infty)$. In particular, $E(x)$ is continuous everywhere.

Proof First we establish convergence. For $x \in [-a, a]$, we use the estimate (1) in the proof of Theorem 12.1: $|x|^k/k! \leq M/2^k$ for $k \geq k_0 + 1$. Therefore, for the partial sums s_n and s_m,$m, n \geq k_0 + 1, m \geq n$,

$$\begin{aligned}
\Big| s_m(x) - s_n(x) \Big| &= \sum_{k=n+1}^{m} \frac{|x|^k}{k!} \\
&\leq M \sum_{k=n_1}^{m} \frac{1}{2^k} \\
&\leq \frac{M}{2^{n+1}} \sum_{k=0}^{\infty} 2^{-k} \\
&= \frac{M}{2^n} \ ,
\end{aligned}$$

which shows that $s_n(x)$ is a Cauchy sequence, hence $s(x)$ converges.

To show that s is continuous, we let $m \to \infty$ in the estimate above to get

$$\Big| s(x) - s_n(x) \Big| \leq \frac{M}{2^n} \ .$$

Let $c \in (-a, a)$. Given $\varepsilon > 0$, fix n such that $M/2^n < \varepsilon/3$. Then

$$|s(x) - s(c)| \leq |s(x) - s_n(x)| + |s_n(x) - s_n(c)| + |s_n(c) - s(c)| \leq \varepsilon/3 + |s_n(x) - s_n(c)| + \varepsilon/3 \ .$$

We fix n and observe that s_n is a polynomial and hence is continuous at c. Hence we can find some δ such that $|s_n(x) - s_n(c)| < \varepsilon/3$ for $x \in (-a, a), |x - c| < \delta$. It follows that

$$|s(x) - s(c)| < \varepsilon,$$

that is, s is continuous at c.